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Cryptoreality of nonanticommutative Hamiltonians

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Abstract: We note that, though nonanticommutative (NAC) deformations of Minkowski supersymmetric theories do not respect the reality condition and seem to lead to non-Hermitian Hamiltonians H, the latter belong to the class of "cryptoreal" Hamiltonians considered recently by Bender and collaborators. They can be made manifestly Hermitian via the similarity transformation $H \to e^R H e^{-R}$ with a properly chosen R. The deformed model enjoys *the same* supersymmetry algebra as the undeformed one, though being realized differently on the involved canonical variables. Besides quantum-mechanical models, we treat, along similar lines, some NAC deformed field models in $4D$ Minkowski space.

Keywords: Non-Commutative Geometry, Superspaces.

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1. Introduction

Supersymmetric models with nonanticommutative (NAC) deformations [1] have recently attracted a considerable interest. The main idea is that the odd superspace coordinates θ^α and $\bar{\theta}^{\dot{\alpha}}$ are not treated as strictly anticommuting anymore, but involve non-vanishing anticommutators [2].[1] In original Seiberg's paper and in many subsequent works (see e.g. [3, 4] and references therein), the deformation is performed in Euclidean rather than Minkowski space-time. The reason is that in Minkowski space it seems impossible to preserve both supersymmetry and reality of the action after deformation, still retaining simple properties of the corresponding $\star$-product (e.g., associativity and nilpotency) [5]. As discussed in [1], Euclidean NAC theories are of interest in stringy perspectives.[2] An interesting question is whether NAC theories are meaningful by themselves, leaving aside the issue of their relationships with string theory. In other words — whether it is possible to consistently define them in Minkowski space, introduce a Hamiltonian with real spectrum and find a unitary evolution operator.

We argue that the answer to this question is positive. Our consideration is mostly based on the analysis of an interesting 1-dimensional NAC model constructed in a recent paper of Aldrovandi and Schaposnik [8]. In that work, NAC deformations of the conventional Witten's supersymmetric quantum mechanics (SQM) model [9] were studied in the chiral basis. In this case, the deformation operator commutes with the supercharge Q, but does not commute with $\bar{Q}$. However, Aldrovandi and Schaposnik noticed the presence of the second supercharge $\bar{\mathcal{Q}}$ that commutes with the Hamiltonian. On the other hand, Q and $\bar{\mathcal{Q}}$ seem not to be Hermitian conjugate to each other and the deformed Hamiltonian also seemingly lacks the Hermitian property.

[1]In other words, the original Grassmann algebra of the odd coordinates is deformed into a *Clifford* algebra.

[2]The stringy origin of such deformations [6] was actually the main motivation of their consideration in [1] (see also [7, 4]).

Our key observation is that, in spite of having a complex appearance, this Hamiltonian is actually Hermitian in disguise. One can call it "crypto-Hermitian" (or "cryptoreal"). It belongs to the class of Hamiltonians studied recently by Bender and collaborators [10]. The simplest example is

$$H = \frac{p^2 + x^2}{2} + igx^3 \,. \tag{1.1}$$

In spite of the manifestly complex potential, it is possible to endow the Hamiltonian (1.1) with a properly defined Hilbert space such that the spectrum of H is real. The clearest way to see this is to observe the existence of the operator R such that the conjugated Hamiltonian

$$\tilde{H} = e^R H e^{-R} \tag{1.2}$$

is manifestly self-adjoint [11]. The explicit form of R for the Hamiltonian (1.1) is[3]

$$R = g\left(\frac{2}{3}p^3 + x^2 p\right) - g^3\left(\frac{64}{15}p^5 + \frac{20}{3}p^3x^2 + 4px^4 - 6p\right) + O(g^5) \,. \tag{1.3}$$

The rotated Hamiltonian is

$$\tilde{H} = \frac{p^2 + x^2}{2} + g^2\left(3p^2x^2 + \frac{3x^4}{2} - \frac{1}{2}\right) + O(g^4) \,. \tag{1.4}$$

The (real) spectrum of $\tilde{H}$ (and H) can be found to any order in g in the perturbation theory, and also non-perturbatively.

We will see that in the case of the Aldrovandi-Schaposnik Hamiltonian, there also exists the operator R making the Hamiltonian Hermitian. The rotated supercharges $e^R Q e^{-R}$ and $e^R \bar{Q} e^{-R}$ are Hermitian-conjugated.

We start in section 2 by constructing the operator R for certain non-supersymmetric Hamiltonians. In particular, we discuss holomorphic deformations (adding to the Hamiltonian a holomorphic function of a complex dynamic variable). In section 3, we present the Aldrovandi-Schaposnik model, find the corresponding operator R, as well as the rotated Hamiltonian and supercharges. Also we briefly consider a NAC deformation of the SQM model with two sorts of chiral supermultiplets. In section 4, we discuss possible generalizations to field theory.

2. Cryptoreality: some comments

- First, about the term "cryptoreality". In the original papers [10], the Hermiticity of the Hamiltonian (1.1) and its relatives was deduced from a certain special symmetry of this Hamiltonian, the $\mathcal{PT}$-symmetry. Indeed, the Hamiltonian (1.1) is invariant with respect to the combination of the parity transformation (which changes the

[3] Actually, what is written here is the Weyl symbol of the operator R. The expression for a contribution to the quantum operator corresponding to a monomial $\sim p^n x^n$ in its Weyl symbol is a properly symmetrized structure, $px \to (1/2)(\hat{p}x + x\hat{p})$, $x^2 p \to (1/3)(x^2\hat{p} + \hat{p}x^2 + x\hat{p}x)$, etc.

sign of x) and the time reversal transformation (which changes i to $-i$). The $\mathcal{PT}$-symmetry of the Hamiltonian might be a sufficient condition for the existence of the operator R such that the conjugated Hamiltonian (1.2) is manifestly Hermitian, but, as we will see later, it is not a necessary condition. In ref. [11], the term "pseudo-Hermiticity" was used. To our mind, however, there is nothing "pseudo" about it, the Hamiltonian (1.1) is simply Hermitian (in the properly defined Hilbert space), but its Hermiticity is hidden, not immediately obvious. That is why the term "crypto-Hermiticity" (or "cryptoreality") seems to us somewhat more appropriate.

- The conjugation (1.2) acts upon all operators including the operators p, x. The Weyl symbols of the transformed operators p', x' are

$$\begin{aligned} p' &= p + 2igxp + g^2(2p^3 - px^2) + \cdots \\ x' &= x - ig(x^2 + 2p^2) - g^2(x^3 - 2xp^2) + \cdots . \end{aligned} \tag{2.1}$$

One can actually obtain the expression (1.4) for the Weyl symbol of the rotated Hamiltonian by simply expressing H in terms of p', x'. The commutator $[p, x]$ is not changed after conjugation, that means that the *Moyal bracket* $\{p', x'\}_{M.B.}$ is equal to one. The Moyal bracket is defined as [12]

$$\{A, B\}_{M.B.} \;=\; 2 \sin \left[\frac{1}{2} \left(\frac{\partial^2}{\partial p \partial X} - \frac{\partial^2}{\partial P \partial x} \right) \right] A(p, x) B(P, X) \Big|_{p=P, x=X} \tag{2.2}$$

The expansion starts with the Poisson bracket, but, generically, there are also higher terms. In particular, $\{p', x'\}_{M.B.}$ differs from $\{p', x'\}_{P.B.}$ by the terms of order $\sim g^4$ and higher. But that means that (2.1) *is* not a canonical transformation. And this means that the *classical* dynamics of $H(p, x)$ and $H(p', x')$ are different. The quantum dynamics of the original and conjugated Hamiltonians is, however, the same.

- One can rotate away not only imaginary pieces in the potential, but also other unfriendly looking terms in the Hamiltonian. For example, one can consider the Hamiltonian

$$H \;=\; \frac{p^2 + x^2}{2} + gx^3 \tag{2.3}$$

and conjugate it with the operator R coinciding with the expression in eq. (1.3) multiplied by the factor $-i$. The conjugated Hamiltonian coincides with (1.4), with the sign of g^2 being reversed. The spectrum of the Hamiltonian (2.3) can be found by the same token as for the Hamiltonian (1.1). Actually, an exact mapping relating the system (2.3) to the system (1.1) exists. Indeed, for any eigenfunction $\Psi_n(x)$ of the Hamiltonian (1.1) with eigenvalue E_n, the function $\Psi_n(-ix)$ is an eigenfunction of the Hamiltonian (2.3) with the eigenvalue $-E_n$.

The appearance of complex values of x may be somewhat unusual, but it is actually an inherent feature of the crypto-Hermitian systems. The eigenfunctions of the Hamiltonian are required to behave well (be not singular and die out for large absolute values of the argument) in a certain domain in the complex x-plane that might or

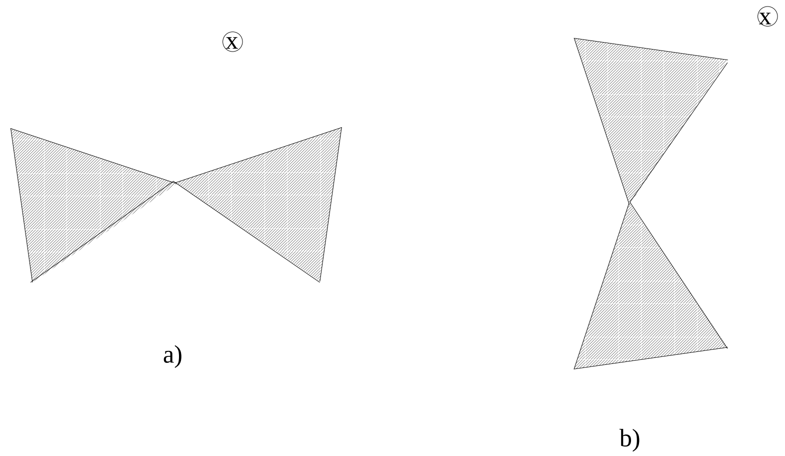

Figure 1: Wave functions for *a)* the Hamiltonian (1.1) and *b)* Hamiltonian (2.3) asymptotically die out in the dashed sectors.

might not include the real axis [10]. The relevant domains for the Hamiltonians (1.1) and (2.3) are shown in figure 1. One is rotated with respect to the other by the angle $\pi/2$.

Another unusual feature of the Hamiltonian (2.3) is the absence of the ground state - the state with the lowest energy. In this case, the spectrum has an upper rather than lower bound. But the overall sign of energy is in fact a matter of book-keeping. For all physical purposes, the dynamics of the Hamiltonian (1.1) in the region in figure 1a and the dynamics of the Hamiltonian (2.3) in the region in figure 1b are equivalent.

Consider now the Hamiltonian

$$H = \bar{\pi}\pi + \bar{z}z + gz^3 \,. \tag{2.4}$$

Remarkably, by conjugating it with the operator

$$R = -ig\left(\bar{\pi}z^2 + \frac{2}{3}\bar{\pi}^3\right), \tag{2.5}$$

one can rotate away the cubic term in the potential *without trace* such that the conjugated Hamiltonian $H' = \bar{\pi}'\pi' + \bar{z}'z' + gz'^3$ is simply $\bar{\pi}\pi + \bar{z}z$. Hence the spectrum of the Hamiltonian (2.4) coincides with the spectrum of a 2-dimensional oscillator, $E_{n,m} = 1 + n + m$. The wave functions of the original Hamiltonian (2.4) are obtained from the oscillator wave functions by conjugation $\Psi = e^{-R}\tilde{\Psi}$. For example, the ground state wave function is

$$\Psi_0 \sim \exp\left\{-\frac{gz^3}{3} - \bar{z}z\right\}. \tag{2.6}$$

It decays exponentially in the three sectors in the complex plane of z shown in figure 2, and the Hilbert space where the crypto-Hermitian Hamiltonian (2.4) is well defined is formed by the functions sharing this property.

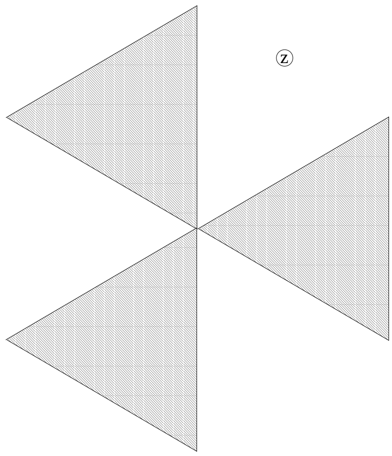

Figure 2: The same for the Hamiltonian (2.4).

By the same token, one can rotate away without trace *any* holomorphic term in the potential. For example, for the Hamiltonian $\bar{\pi}\pi + \bar{z}z + gz^5$, this is done with the operator

$$R = -ig\left(\bar{\pi}z^4 + \frac{4}{3}\bar{\pi}^3 z^2 + \frac{8}{15}\bar{\pi}^5\right).$$

Generally, the operator rotating away the term gz^N in the potential has the form

$$R_N = -ig\bar{\pi}z^{N-1} f_N\left(\frac{\bar{\pi}}{z}\right),$$

with $f_N(r)$ satisfying the equation

$$[1 - r^2(N-1)]f_N + r(1+r^2)f'_N = 1\ . \tag{2.7}$$

When N is odd, the solution represents a polynomial. For even N, it is more complicated. For example,

$$f_2(r) = \frac{1}{2}\left[\frac{1+r^2}{r}\arctan r + 1\right]. \tag{2.8}$$

- Cryptoreal Hamiltonians for the systems with continuum number of degrees of freedom also exist. Bender, Brody, and Jones found the proper conjugation operator for the system described by the Lagrangian [10]

$$\mathcal{L} \ = \ \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\mu^2\phi^2}{2} - ig\phi^3\,, \tag{2.9}$$

ϕ is a real scalar field. In the lowest order in g, it is given by a nonlocal expression

$$R \ = \ \iiint d\mathbf{x}d\mathbf{y}d\mathbf{z}\,[M_{\mathbf{xyz}}p_{\mathbf{x}}p_{\mathbf{y}}p_{\mathbf{y}} + N_{\mathbf{xyz}}\phi_{\mathbf{x}}\phi_{\mathbf{y}}p_{\mathbf{z}}]\,, \tag{2.10}$$

where $p_{\mathbf{x}}$ are canonical momenta, $p_{\mathbf{x}} = -i\partial/\partial\phi_{\mathbf{x}}$, and the kernels $M_{\mathbf{xyz}}, N_{\mathbf{xyz}}$ have a complicated, but explicit form.

We want to notice that the system of the complex scalar field φ with the interaction Hamiltonian $\sim \varphi^3$ is also cryptoreal, and the corresponding conjugation operator is given, again, by the expression (2.10) with $\bar{\pi}_{\mathbf{x}} = -i\partial/\partial\bar{\varphi}_{\mathbf{x}}$ being substituted for $p_{\mathbf{x}}$. This operator rotates the interaction term away without trace by the same token as the operator (2.5) rotates it away in the QM case.

Actually, the pattern is quite general. Any holomorphic interaction term can be entirely rotated away simply because the proper conjugation operator R involves in this case only the momentum operators $\bar{\pi}_{\mathbf{x}}$ rather than $\pi_{\mathbf{x}}$, and $\bar{\partial} f = 0$ for holomorphic functions.

- Finally, let us reproduce here the arguments of [10] displaying the reality of the spectrum of a $\mathcal{PT}$-symmetric Hamiltonian. The operator $\mathcal{PT}$ commutes with the Hamiltonian, and it is reasonable to assume that a basis of the states representing the eigenstates of both $\mathcal{PT}$ and H can be chosen.[4] Let Ψ be an eigenstate of both $\mathcal{PT}$ and H,

$$\mathcal{PT}\,\Psi \;=\; \lambda\Psi, \qquad H\,\Psi \;=\; E\Psi\,. \tag{2.11}$$

Applying the operator $\mathcal{PT}$ to the second equality and using $[\mathcal{PT}, H] = 0$ and $\mathcal{PT}(E\Psi) = E^*\mathcal{PT}(\Psi)$, we conclude that $E = E^*$ *Q.E.D.* Note also that applying $\mathcal{PT}$ to the first equality and using $(\mathcal{PT})^2 = 1$, one can show that $\lambda\lambda^* = 1$ and hence $\lambda = e^{i\alpha}$. By going from Ψ to $\Psi e^{-i\alpha/2}$, one can set $\lambda = 1$.

The norm of some eigenstates may happen to be negative. However, this can be mended [10] if redefining inner product by including in its definition the action of the "charge conjugation" operator $\mathcal{C}$ that commutes with both H and $\mathcal{PT}$ and is defined as

$$\mathcal{C}(x,y) \;=\; \sum_n \Psi_n(x)\Psi_n^*(y)\,. \tag{2.12}$$

The operator $\mathcal{C}$ is in fact closely related to the operator R rotating the Hamiltonian to the manifestly Hermitian form, as discussed above,

$$\mathcal{C} \;=\; e^{-2R}\mathcal{P}\,. \tag{2.13}$$

3. Aldrovandi-Schaposnik model

The simplest SQM model [9] involves a real supervariable

$$X(\theta,\bar{\theta},t) \;=\; x(t) + \theta\psi(t) + \bar{\psi}(t)\bar{\theta} + \theta\bar{\theta}F(t)\,. \tag{3.1}$$

The action is

$$S \;=\; -\int dt\, d^2\theta \left[\frac{1}{2}(DX)(\bar{D}X) + V(X)\right], \tag{3.2}$$

[4] Were $\mathcal{PT}$ a linear operator, it would be trivial, but $\mathcal{PT}$ involves complex conjugation and is not linear. Hence, the existence of such basis is, indeed, an *assumption* and the reasoning given here cannot be regarded as a formal proof.

with the convention $\int d^2\theta\, \theta\bar{\theta} = 1$. Here $V(X)$ is the superpotential and $D, \bar{D}$ are covariant derivatives. Bearing in mind the deformation coming soon, we will choose their left chiral basis representation

$$D = \frac{\partial}{\partial \theta} - 2i\bar{\theta}\frac{\partial}{\partial t}, \qquad \bar{D} = -\frac{\partial}{\partial \bar{\theta}} . \tag{3.3}$$

Here $t = \tau - i\theta\bar{\theta}$ and τ is the real time coordinate of the central basis. Asymmetry between D and $\bar{D}$ makes the Lagrangian following from (3.2) complex,

$$L = -i\dot{x}F - \frac{\partial V(x)}{\partial x}F + \frac{1}{2}F^2 + i\bar{\psi}\dot{\psi} + \frac{\partial^2 V(x)}{\partial x^2}\bar{\psi}\psi , \tag{3.4}$$

but one can easily make it real, rewriting it in terms of $\tilde{F} = F - i\dot{x}$ and subtracting a total derivative. This corresponds to going over to the central basis from the chiral one.

The deformation is introduced by postulating non-vanishing anticommutators

$$\{\theta, \theta\} = C, \qquad \{\bar{\theta}, \bar{\theta}\} = \bar{C}, \qquad \{\theta, \bar{\theta}\} = \tilde{C} . \tag{3.5}$$

The deformed action involves star products,

$$S = -\int dt\, d^2\theta \left[\frac{1}{2}(D \star X) \star (\bar{D} \star X) + V_\star(X)\right], \tag{3.6}$$

where

$$X \star Y = \exp\left\{-\frac{C}{2}\frac{\partial^2}{\partial\theta_1\partial\theta_2} - \frac{\bar{C}}{2}\frac{\partial^2}{\partial\bar{\theta}_1\partial\bar{\theta}_2} - \frac{\tilde{C}}{2}\left(\frac{\partial^2}{\partial\theta_1\partial\bar{\theta}_2} + \frac{\partial^2}{\partial\bar{\theta}_1\partial\theta_2}\right)\right\} X(1)Y(2)\Big|_{1=2} \tag{3.7}$$

and $V_\star(X)$ is obtained from $V(X) = \sum_n c_n X^n$ by substituting $X^2 \to X_\star^2 \equiv X \star X$, $X^3 \to X_\star^3 \equiv X \star X \star X$, etc in its Taylor expansion. The star product in (3.6) just ensures the Weyl ordering of any product of the θ monomials such that

$$\theta \star \theta = \frac{C}{2}, \quad \bar{\theta} \star \bar{\theta} = \frac{\bar{C}}{2}, \quad \theta \star \bar{\theta} = \theta\bar{\theta} + \frac{\tilde{C}}{2}, \quad \bar{\theta} \star \theta = \bar{\theta}\theta + \frac{\tilde{C}}{2},$$

in accordance with the basic relation (3.5). The star product is associative.

The component expression for the deformed Lagrangian is the same as in eq. (3.4), with $V(x)$ being substituted by [13, 8]

$$\tilde{V}(x, F) = \int_{-1/2}^{1/2} d\xi\, V(x + \xi c F), \tag{3.8}$$

where

$$c^2 = \tilde{C}^2 - C\bar{C} \tag{3.9}$$

is the relevant deformation parameter.[5] If $\bar{C}$ is conjugate to C and $\tilde{C}$ is real, c^2 is also real. Note, however, that one may, generally speaking, lift the condition that θ and $\bar{\theta}$ are

[5]The relation (3.8) can be easily derived by keeping the term $\propto \theta\bar{\theta}$ in the products $X_\star^n$, with using associativity and the identity $(\theta\bar{\theta}) \star (\theta\bar{\theta}) = c^2/4$. Note the correct sign of c^2 in (3.9) as compared to the wrong one in the definition of c^2 in [8].

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conjugate to each other, in which case $C, \bar{C}$ and $\tilde{C}$ can take arbitrary values. We still require the reality of c^2. The crypto-Hermiticity of the deformed Hamiltonian discussed below is fulfilled under this condition.

In the simplest nontrivial case, $V(X) = \lambda X^3/3$,

$$\tilde{V}(x,F) = \frac{\lambda x^3}{3} + \frac{\lambda c^2 x F^2}{12} . \tag{3.10}$$

The corresponding canonical Hamiltonian is

$$H = \frac{p^2}{2} + i\frac{\partial \tilde{V}}{\partial x}p - \frac{\partial^2 \tilde{V}}{\partial x^2}\bar{\psi}\psi , \tag{3.11}$$

with $p = -iF$. The deformed Lagrangian and Hamiltonian look inherently complex. Obviously, the complexities now cannot be removed by simply going from the chiral to the central basis.

In the chiral basis, the supercharges are represented by the following superspace differential operators,

$$Q = \frac{\partial}{\partial\theta} , \qquad \bar{Q} = -\frac{\partial}{\partial\bar{\theta}} - 2i\theta\frac{\partial}{\partial t} . \tag{3.12}$$

Note that the star product operator (3.7) still commutes with Q (in other words, the Leibnitz rule $Q \star (X \star Y) = (Q \star X) \star Y + X \star (Q \star Y)$ still holds), but not with $\bar{Q}$. That means that the deformed action (3.6) is still invariant with respect to the supersymmetry transformations generated by Q, but not $\bar{Q}$. The Q-invariance implies the existence of the conserved Nöther supercharge whose component phase space expression is simply

$$\mathcal{Q} = \psi p . \tag{3.13}$$

As was observed in [8], there is another Grassmann-odd operator commuting with the Hamiltonian. It reads

$$\bar{\mathcal{Q}} = \bar{\psi}\left(p + 2i\frac{\partial \tilde{V}}{\partial x}\right) . \tag{3.14}$$

The standard SUSY algebra

$$\mathcal{Q}^2 = \bar{\mathcal{Q}}^2 = 0, \qquad \{\mathcal{Q}, \bar{\mathcal{Q}}\} = 2H \tag{3.15}$$

holds, but, naively, $\bar{\mathcal{Q}}$ is not adjoint to $\mathcal{Q}$ and H is not Hermitian.

Let us show now that the Hamiltonian (3.11) is in fact cryptoreal. Consider for simplicity only the case (3.10). We have,[6]

$$H = \frac{p^2}{2} + i\lambda p x^2 - i\beta p^3 - 2\lambda x\bar{\psi}\psi , \tag{3.16}$$

where $\beta = \lambda c^2/12$.

[6]Note that this Hamiltonian is not $\mathcal{PT}$-, but just $\mathcal{T}$-symmetric.

It is convenient to treat λ and β on equal footing and to get rid of the complexities $\sim ipx^2$ and $\sim ip^3$ simultaneously. The operator R doing this job is

$$R \;=\; -\frac{\lambda x^3}{3} + \beta x p^2 - 2\lambda\beta x^2 \bar\psi\psi + \cdots\,, \tag{3.17}$$

where the dots stand for the terms of the third and higher order in λ and/or β. The conjugated Hamiltonian is

$$\tilde H = e^R H e^{-R} \;=\; \frac{p^2}{2} - 2\lambda x\bar\psi\psi + \frac{1}{2}[\lambda^2 x^4 + 3\beta^2 p^4] + \frac{1}{2}\lambda\beta + O(\lambda^3, \beta^3, \lambda^2\beta, \lambda\beta^2)\,. \tag{3.18}$$

It is Hermitian. The rotated supercharges are

$$\begin{aligned} \tilde Q &= e^R Q e^{-R} = \psi[p - i(\lambda x^2 - \beta p^2) + \lambda\beta x^2 p - \beta^2 p^3 + \cdots]\,, \\ \tilde{\bar Q} &= e^R \bar Q e^{-R} = \bar\psi[p + i(\lambda x^2 - \beta p^2) + \lambda\beta x^2 p + 3\beta^2 p^3 + \cdots]\,. \end{aligned} \tag{3.19}$$

We observe that they are still not adjoint to each other. To make them mutually adjoint to the considered order in β, λ, one should add to the operator R one more term

$$R \;\Rightarrow\; \hat R = R - 2\beta^2 p^2 \bar\psi\psi\,. \tag{3.20}$$

It is easy to see that this modification does not change the rotated Hamiltonian in the considered order, but ensures the rotated supercharges to be manifestly adjoint to each other

$$\begin{aligned} \hat Q &= e^{\hat R} Q e^{-\hat R} = \psi[p - i(\lambda x^2 - \beta p^2) + \lambda\beta x^2 p + \beta^2 p^3 + \cdots]\,, \\ \hat{\bar Q} &= e^{R} \bar Q e^{-R} = \bar\psi[p + i(\lambda x^2 - \beta p^2) + \lambda\beta x^2 p + \beta^2 p^3 + \cdots]\,. \end{aligned} \tag{3.21}$$

By construction, the operators $\hat Q, \hat{\bar Q}$ and $\tilde H$ satisfy the standard algebra (3.15). We see that the requirement of the mutual adjointness of supercharges is to some extent more fundamental than that of the Hermiticity of the Hamiltonian — the latter does not strictly fix the rotation operator R while the former does.

One can be convinced, order by order in β, λ, that complexities in H can be successfully rotated away also in higher orders (with simultaneously restoring the mutual conjugacy of the supercharges), and this is also true for higher powers $N > 3$ in $V(X) \sim X^N$ and hence for any analytic superpotential.[7]

As the last topic of this section, we shall consider NAC deformations of some other $\mathcal{N} = 1$ SQM models.[8]

Besides the $\mathcal{N} = 1$ multiplet with the off-shell content $(\mathbf{1}, \mathbf{2}, \mathbf{1})$, there also exist chiral $\mathcal{N} = 1$ multiplets $(\mathbf{2}, \mathbf{2}, \mathbf{0})$ and $(\mathbf{0}, \mathbf{2}, \mathbf{2})$, having, correspondingly, even and odd overall Grassmann parity. They are described, respectively, by the chiral superfields $\Phi(\theta, \bar\theta, t)$ and $\Psi(\theta, \bar\theta, t)$:

$$\bar D\Phi = 0 \quad\Rightarrow\quad \Phi = z(t) + \theta\chi(t)\,, \quad \bar D\Psi = 0 \quad\Rightarrow\quad \Psi = \omega(t) + \theta h(t)\,, \tag{3.22}$$

[7] It would be worth being aware of the full analytic proof of this.

[8] We denote by $\mathcal{N}$ the number of complex supercharges.

where as before $t = \tau - i\theta\bar{\theta}$, z is a complex bosonic field, ξ and ω are complex fermionic fields and h is a complex bosonic auxiliary field. It was shown in [8] that the only NAC deformation preserving the $1D$ chirality and anti-chirality corresponds to the choice $\bar{C} = \tilde{C} = c^2 = 0, C \neq 0$ in (3.5). Then the action of Φ,

$$S_\Phi = -\int dt\, d^2\theta \left[\frac{1}{4} D\Phi\bar{D}\bar{\Phi} + K(\Phi, \bar{\Phi})\right] = \int dt \left(\dot{z}\dot{\bar{z}} + \frac{i}{2}\bar{\chi}\dot{\chi} + \cdots\right), \tag{3.23}$$

remains undeformed after replacing all products by the relevant $\star$ products [8].

Actually, the same is true for the action of Ψ

$$S_\Psi = -\int dt\, d^2\theta \left[\frac{1}{4}\Psi\bar{\Psi} + \beta\bar{\theta}\Psi - \bar{\beta}\theta\bar{\Psi}\right] = \int dt \left(\frac{i}{2}\bar{\omega}\dot{\omega} - \frac{1}{4}h\bar{h} + \beta h + \bar{\beta}\bar{h}\right), \tag{3.24}$$

where β is a complex constant. However, while considering mutual couplings of Ψ and Φ, there arise new possibilities. Prior to switching on any deformation, such couplings provide potential terms for the $(\mathbf{2}, \mathbf{2}, \mathbf{0})$ multiplet which do not exist within the pure Φ system (the "potential" term $K(\Phi, \bar{\Phi})$ in (3.23) produces only a Wess-Zumino type term $\sim (\dot{z}\bar{z} - \dot{\bar{z}}z) + \cdots$.). In particular, one can consider the action

$$S_{\Phi+\Psi} = -\int dt\, d^2\theta \left[\frac{1}{4} D\Phi\bar{D}\bar{\Phi} + \frac{1}{4}\Psi\bar{\Psi} + \bar{\theta}\Psi\mathcal{F}(\Phi) - \theta\bar{\Psi}\bar{\mathcal{F}}(\bar{\Phi})\right], \tag{3.25}$$

which gives rise to a non-trivial scalar potential for $z, \bar{z}$ upon eliminating the auxiliary fields $h, \bar{h}$ by their equations of motion. For instance, choosing $\mathcal{F} = a + b\Phi + d\Phi^2$, one obtains after elimination of $h, \bar{h}$ the following on-shell component action

$$S_{\Phi+\Psi} = \int dt \left[|\dot{z}|^2 + \frac{i}{2}(\bar{\chi}\dot{\chi} + \bar{\omega}\dot{\omega}) + 4|a + bz + dz^2|^2 + \text{Yukawa } \omega, \chi \text{ couplings}\right]. \tag{3.26}$$

Nevertheless, once again, the direct (anti)chirality-preserving deformation of (3.25) does not yield nothing new. The reason is that the terms proportional to the deformation constant C either never appear (in the antiholomorphic part $\sim \bar{\Psi}$) or vanish after doing the Berezin integral (in the holomorphic part $\sim \Psi$).

There still exists an interesting mechanism of generating new potential terms via the deformation. It is based on the observation that, while $\Psi^2 = 0$ because of the Grassmann character of Ψ, this nilpotency property is not longer valid for $\Psi \star \Psi$ and higher-order star products. Indeed, we find

$$\Psi \star \Psi = \frac{C}{2}h^2\,,\ \Psi \star (\Psi \star \Psi) = \frac{C}{2}\left(\omega h^2 + \theta h^3\right),\ \Psi \star (\Psi \star \Psi \star \Psi) = \frac{C^2}{4}h^4\,, \quad \text{etc}. \tag{3.27}$$

The star products of $\bar{\Psi}$ coincide with the ordinary ones and so are identically zero. Let us then e.g. add to the Lagrangian in (3.25), with $\mathcal{F} = a + bz$ as the simplest choice, the term $a_1\, \bar{\theta}\Psi \star (\Psi \star \Psi)$. The bosonic part of the corresponding component Lagrangian is given by the following expression

$$L = |\dot{z}|^2 - \frac{1}{4}h\bar{h} + h(a + bz) + \bar{h}(\bar{a} + \bar{b}\bar{z}) - \frac{a_1 C}{2}h^3\,. \tag{3.28}$$

Here we cannot longer treat $\bar{h}$ as a conjugate of h: both these fields should now be treated as independent complex ones. Eliminating $\bar{h}$ by its equation of motion, we obtain

$$L = |\dot{z}|^2 + 4|a + bz|^2 - 32a_1 C(\bar{a} + \bar{b}\bar{z})^3 \, . \tag{3.29}$$

The additional term is holomorphic; by the same token as in section 2 we conclude that the corresponding term in the quantum Hamiltonian can be rotated away without trace! So the modified system proves to be physically equivalent to the undeformed system (has the same quantum spectrum) in spite of an apparent difference in their Lagrangians.

The star product deformation breaks a half of supersymmetries and the modified action is manifestly invariant only under the holomorphic half of the original supersymmetry. Since after rotation we reproduce the original system, the modified system should also respect some additional hidden supersymmetry of the opposite holomorphy, like in the $(\mathbf{1}, \mathbf{2}, \mathbf{1})$ system of ref. [8] discussed above.

4. Field theories

The first example of an anticommutative deformation of a supersymmetric field theory was considered in ref. [1]. Seiberg took the standard Wess-Zumino model

$$\mathcal{L} \; = \; \int d^4\theta \, \bar{\Phi}\Phi + \left[\int d^2\theta \left(\frac{m\Phi^2}{2} + \frac{\lambda\Phi^3}{3} \right) + \text{c.c} \right] \tag{4.1}$$

(where now $\int d^2\theta \, (\theta^\alpha \theta_\alpha) = 1$) and deformed it by introducing the nontrivial anticommutator

$$\{\theta^\alpha, \theta^\beta\} \; = \; C^{\alpha\beta} \, , \tag{4.2}$$

$C^{\alpha\beta} = C^{\beta\alpha}$, in the assumption that all other (anti)commutators vanish,

$$\{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = \{\theta^\alpha, \bar{\theta}^{\dot{\beta}}\} = [\theta^\alpha, x^L_\mu] = [\bar{\theta}^{\dot{\alpha}}, x^L_\mu] = [x^L_\mu, x^L_\nu] = 0 \; . \tag{4.3}$$

Note that this all was written in the *chiral* basis, $x^L_\mu = x^{\text{central}}_\mu + i\theta\sigma_\mu\bar{\theta}$. In ref. [1], the space x_μ was assumed to be Euclidean. We will work in Minkowski space, however, and will not be scared by the appearance of complexities at intermediate steps. The Minkowski space deformation (4.2), (4.3) is analogous to the SQM deformation (3.5) with $\bar{C} = \tilde{C} = 0$.

The anticommutator (4.2) introduces a constant self-dual tensor which explicitly breaks Lorentz invariance. However, the deformed Lagrangian expressed in terms of the component fields proves still to be Lorentz invariant. Indeed, it is easy to find that the kinetic term $\int d^4\theta \, \bar{\Phi}\Phi$ is undeformed and the only extra piece comes from

$$\Delta\mathcal{L} = \frac{\lambda}{3} \int d^2\theta \, \Phi * \Phi * \Phi - \frac{\lambda}{3} \int d^2\theta \, \Phi^3 = -\frac{\lambda}{3} \, \det \|C\| F^3 \; . \tag{4.4}$$

It depends only on the scalar $\det \|C\|$ and is obviously Lorentz invariant. Adding the usual terms $F(m\phi + \lambda\phi^2) + \bar{F}(m\bar{\phi} + \lambda\bar{\phi}^2)$ coming from superpotential and $F\bar{F}$ from the

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kinetic term, and expressing F and $\bar{F}$ via ϕ and $\bar{\phi}$, we see that the undeformed potential $|m\phi + \lambda\phi^2|^2$ acquires an extra holomorphic contribution $\sim (m\bar{\phi} + \lambda\bar{\phi}^2)^3$.

We have seen, however, that such a holomorphic deformation can be rotated away without trace! In other words, the deformation (4.2) does not change the dynamics (the spectrum of the Hamiltonian etc) of the Wess-Zumino model in Minkowski space.[9]

The final example is the deformed $\mathcal{N} = 2$ gauge theory [7, 14]. There exists in this case a natural Lorentz invariant deformation [7],

$$\{\theta_i^\alpha, \theta_j^\beta\} \;=\; \frac{1}{4} J \epsilon^{\alpha\beta} \epsilon_{ij} \,, \tag{4.5}$$

$i, j = 1, 2$.[10] The Lagrangian of the deformed $\mathcal{N} = 2$ supersymmetric U(1) theory is [7, 14]

$$\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_\Psi + \mathcal{L}_A \,, \tag{4.6}$$

$$\mathcal{L}_\phi = -\frac{1}{2} \Box \bar{\phi} \left[\phi + \frac{J A_m A_m}{1 + J\bar{\phi}} + \frac{1}{4} \frac{J^3 \partial_m \bar{\phi} \partial_m \bar{\phi}}{1 + J\bar{\phi}} \right] , \tag{4.7}$$

$$\mathcal{L}_\Psi = i \left[\Psi^{i\alpha} + \frac{J A_m (\sigma_m)^\alpha_{\dot{\alpha}} \bar{\Psi}^{i\dot{\alpha}}}{1 + J\bar{\phi}} \right] (\sigma_n)_{\alpha\dot{\beta}} \, \partial_n \left(\frac{\bar{\Psi}_i^{\dot{\beta}}}{1 + J\bar{\phi}} \right) , \tag{4.8}$$

$$\mathcal{L}_A = \frac{1}{4} (1 + J\bar{\phi})^2 \left(f_{mn} f_{mn} + f_{mn} \tilde{f}_{mn} \right) , \tag{4.9}$$

$$f_{mn} = \partial_m \left(\frac{1}{1 + J\bar{\phi}} A_n \right) - \partial_n \left(\frac{1}{1 + J\bar{\phi}} A_m \right) . \tag{4.10}$$

The Lagrangian (4.6) was derived originally in Euclidean space. In Minkowski space, it is clearly complex. Bearing in mind the previous discussion, it is natural to suggest, however, that the corresponding Hamiltonian is cryptoreal. Leaving the issue of cryptoreality of the full field theory Hamiltonian for future study, let us disregard the fermion part of (4.6) and consider the $1D$ reduction of what is left. We will show that the resulting quantum-mechanical model is cryptoreal and actually amounts to the free model. The reduction goes as

$$\Box \to \partial_t^2 \,, \; \partial_m \phi \partial_m \bar{\phi} \to \dot{\phi} \dot{\bar{\phi}} \,, \; A_n A_n \to -A_0 A_0 + \vec{A}\vec{A} \tag{4.11}$$

and we obtain

$$\mathcal{L}^{QM}_{\rm bos} = \frac{1}{2} \dot{\phi} \dot{\bar{\phi}} + \frac{1}{2} \dot{\vec{A}} \dot{\vec{A}} - \frac{1}{24} \frac{J^4 (\dot{\bar{\phi}})^4}{(1 + J\bar{\phi})^2} + \frac{J}{2} \frac{\ddot{\bar{\phi}}}{1 + J\bar{\phi}} A_0^2 \,. \tag{4.12}$$

The corresponding canonical Hamiltonian, in the obvious notation, is as follows

$$H = 2P\bar{P} + \frac{1}{2} \vec{P}\vec{P} + \frac{2}{3} J^4 \frac{P^4}{(1 + J\bar{\phi})^2} - 2J^2 A_0^2 \frac{P^2}{(1 + J\bar{\phi})^2} \,. \tag{4.13}$$

[9]To avoid a misunderstanding, we would like to point out that even in Minkowski space, the fields ϕ and $\bar{\phi}$ (as well as F and $\bar{F}$) after deformation should be treated as complex fields which *are not conjugate* to each other. The standard complex conjugacy requirements can be consistently imposed on the *rotated* fields and their canonical momenta.

[10]The deformation parameter J is related to the original one I [7] as $J = 4I$.

Making the rotation

$$H' = e^R H e^{-R}\,, \quad R = -\frac{i}{3}\frac{J^3 P^3}{1+J\bar{\phi}} + iA_0^2 \frac{JP}{1+J\bar{\phi}}\,, \tag{4.14}$$

we find that

$$H' = 2P\bar{P} + \frac{1}{2}\vec{P}\vec{P}\,, \tag{4.15}$$

i.e. the deformation is rotated away without trace, like in the examples above, and our quantum-mechanical system is reduced to the free one. In the full 4-dimensional case the situation is more subtle due to the presence of the term $\sim \varepsilon_{mnrq}$ that vanished after reduction. Our simple $1D$ consideration shows that the corresponding dynamics in Minkowski space is expected to be "almost trivial". Nevertheless, we do not see reasons why the deformation in this case can be entirely rotated away. Rather, the situation should be similar to what we observed in the Aldrovandi-Schaposnik model. To get a deeper insight into these issues, it would be instructive to analyze, from a similar point of view, the deformations of the nonabelian $\mathcal{N} = 2$ gauge theories [7] and the models involving hypermultiplets [15], which are not free in the undeformed limit $J = 0$.

5. Discussion

Our main result is that NAC deformations of supersymmetric theories are well defined not only in Euclidean, but also in Minkowski space. In spite of its unfriendly looking complex appearance, the deformed theory can be endowed with a Hilbert space where the Hamiltonian is Hermitian and its spectrum is real. In many cases (in particular in the case of the deformed Wess-Zumino model considered in Seiberg's original paper), the deformed Hamiltonian is actually physically equivalent to the undeformed one. Extra contributions stemming from nonanticommutativity have holomorphic structure and can be "rotated away" without trace, as was explained in the text of the paper. For some other NAC theories, the new Hamiltonian is not equivalent to the old one and deformation brings about nontrivial changes in dynamics.

We discussed at length a one-dimensional SQM example due to Aldrovandi and Schaposnik. While going to $4D$ field theories, the requirement that Lorentz invariance is kept after deformation dictates the undeformed theory to possess at least $\mathcal{N} = 2$ supersymmetry [see eq. (4.5)]. The Lagrangian of the deformed $\mathcal{N} = 2$ gauge theory was constructed before. We have not proven, but argued that it is cryptoreal (i.e. the Hamiltonian can be made Hermitian) but is not equivalent to the undeformed Lagrangian. A thorough study of this interesting question is a problem for the future.

Another interesting direction of study, not related to nonanticommutativity, but related to cryptoreality is the following. In ref. [16], we constructed a gauge theory in six dimensions which is superconformal at the classical level. It is renormalizable, and the variant of the theory involving interaction with a hypermultiplet [17] is anomaly free [18]. However, this theory involves higher derivatives, which may in principle lead to the loss of unitarity due to the presence of ghosts. In particular, the theory involves scalar fields D

of canonical dimension 2 with the potential $\sim D^3$. Naively, such a potential means vacuum instability and the associated loss of unitarity. We have seen, however, that the QM models with the potentials $V(x) \sim ix^3$ or $V(x) \sim x^3$ *can* be meaningful since their Hamiltonians can be made Hermitian. It is not excluded that this is also the case for certain higher-derivative field theories and, in particular, for the models constructed in [16, 17].

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